

BOUNDEDNESS AND EXPONENTIAL CONVERGENCE OF A CHEMOTAXIS MODEL FOR TUMOR INVASION

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ABSTRACT. We revisit the following chemotaxis system modeling tumor invasion

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v), & x \in \Omega, t > 0, \\ v_t = \Delta v + wz, & x \in \Omega, t > 0, \\ w_t = -wz, & x \in \Omega, t > 0, \\ z_t = \Delta z - z + u, & x \in \Omega, t > 0, \end{cases}$$

in a smooth bounded domain $\Omega \subset \mathbb{R}^n (n \geq 1)$ with homogeneous Neumann boundary and initial conditions. This model was recently proposed by Fujie *et al.* [3] as a model for tumor invasion with the role of extracellular matrix incorporated, and was analyzed by Fujie *et al.* [4], showing the uniform boundedness and convergence for $n \leq 3$. In this work, we first show that the L^∞ -boundedness of the system can be reduced to the boundedness of $\|u(\cdot, t)\|_{L^{\frac{n}{4}+\epsilon}(\Omega)}$ for some $\epsilon > 0$ alone, and then, for $n \geq 4$, if the initial data $\|u_0\|_{L^{\frac{n}{4}}}$, $\|z_0\|_{L^{\frac{n}{2}}}$ and $\|\nabla v_0\|_{L^n}$ are sufficiently small, we are able to establish the L^∞ -boundedness of the system. Furthermore, we show that boundedness implies exponential convergence with explicit convergence rate, which resolves the open problem left in [4]. More precisely, it is shown, if $u_0 \geq \not\equiv 0$, then any bounded solution (u, v, w, z) of the tumor invasion model satisfies the following exponential decay estimate:

$$\begin{aligned} & \left\| (u(\cdot, t), v(\cdot, t), w(\cdot, t), z(\cdot, t)) - (\bar{u}_0, \bar{v}_0 + \bar{w}_0, 0, \bar{u}_0) \right\|_{L^\infty(\Omega)} \\ & \leq C \left(e^{-\frac{1}{2} \min\{\lambda_1, \frac{\bar{u}_0}{3}\}t}, e^{-\min\{\lambda_1, \frac{\bar{u}_0}{3}\}t}, e^{-\frac{\bar{u}_0}{2}t}, e^{-\min\{1, \frac{\lambda_1}{2}, \frac{\bar{u}_0}{6}\}t} \right) \end{aligned}$$

for all $t > 0$ and for some constant $C > 0$ independent of time t . Here, for a generic function f , \bar{f} means the spatial average of f over Ω and $\lambda_1(> 0)$ is the first nonzero eigenvalue of $-\Delta$ in Ω with homogeneous Neumann boundary condition.

1. INTRODUCTION AND MAIN RESULTS

In this paper, we shall revisit the chemotaxis system modeling tumor invasion:

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v), & x \in \Omega, t > 0, \\ v_t = \Delta v + wz, & x \in \Omega, t > 0, \\ w_t = -wz, & x \in \Omega, t > 0, \\ z_t = \Delta z - z + u, & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial z}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ (u(x, 0), v(x, 0), w(x, 0), z(x, 0)) = (u_0(x), v_0(x), w_0(x), z_0(x)), & x \in \Omega, \end{cases} \quad (1.1)$$

where $\Omega \subset \mathbb{R}^n (n \geq 1)$ a bounded domain with smooth boundary $\partial\Omega$, $\frac{\partial}{\partial \nu}$ represents the differentiation with respect to the outward normal vector ν of $\partial\Omega$ and the prescribed initial data satisfy

$$(u_0, v_0, w_0, z_0) \in C^0(\bar{\Omega}) \times W^{1,\infty}(\Omega) \times C^1(\bar{\Omega}) \times C^0(\bar{\Omega}) \text{ with } u_0, v_0, w_0, z_0 \geq 0. \quad (1.2)$$

The system (1.1) was recently proposed by Fujie *et al.* [3] as a modified tumor invasion model with chemotaxis effect of Chaplain and Anderson type [1]. According to the cancer

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phenomena point of view, the unknown functions u, v, w and z denote the molar concentration of tumor cells, *active extracellular matrix* (ECM*), extracellular matrix (ECM) and matrix degrading enzymes (MDE), respectively. A particular core of the model is to account for a chemoattractant induced by an ECM*, which is produced by a biological reaction between ECM and MDE. We refer to [3] for more explanations and biological background.

Compared with the *haptotaxis* (in which attractants are non-diffusible) type model usually used to describe cancer invasion as pointed out in [4], the cross-diffusion is of *chemotaxis* type in that it is oriented toward the higher concentration of the diffusible ECM*, the latter being produced by the static ECM together with the chemical MDE. In contrast to the direct Keller-Segel prototypical model of chemotaxis process [8],

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla z), & x \in \Omega, t > 0, \\ z_t = \Delta z - z + u, & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial z}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ (u(x, 0), z(x, 0)) = (u_0(x), z_0(x)), & x \in \Omega, \end{cases} \quad (1.3)$$

which has the possibility of blow-up in a finite/infinite time depending strongly on the space dimension (No blow-up in 1-D [13, 6], critical mass blow-up in 2-D [5, 7, 11, 12, 14, 10] and, generic blow-up in ≥ 3 -D [17, 18]), the indirect chemotaxis mechanism in (1.1) has been shown to have a role in enhancing the regularity and boundedness properties of solutions [4]. Indeed, therein, they showed the boundedness and uniform convergence of (1.1) to a certain spatially constant equilibrium for $n \leq 3$. More precisely, they proved:

(R1) **(Boundedness in ≤ 3 -D)** Let $n \leq 3$ and (1.2) hold. Then there exists a uniquely determined quadruple (u, v, w, z) of nonnegative functions defined on $\bar{\Omega} \times [0, \infty)$ which solves (1.1) classically and is bounded in the sense there exists a constant $C > 0$ such that for all $t > 0$

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,\infty}(\Omega)} + \|w(\cdot, t)\|_{L^\infty(\Omega)} + \|z(\cdot, t)\|_{L^\infty(\Omega)} < C. \quad (1.4)$$

(R2) **(Uniform Convergence)** Let $n \leq 3$ and suppose that (1.2) hold and $u_0 \geq \not\equiv 0$. Then the solution of (1.1) fulfills

$$\left\| (u(\cdot, t), v(\cdot, t), w(\cdot, t), z(\cdot, t)) - (\bar{u}_0, \bar{v}_0 + \bar{w}_0, 0, \bar{u}_0) \right\|_{L^\infty(\Omega)} \rightarrow 0 \quad (1.5)$$

as $t \rightarrow \infty$, where

$$\bar{u}_0 = \frac{1}{|\Omega|} \int_{\Omega} u_0, \quad \bar{v}_0 = \frac{1}{|\Omega|} \int_{\Omega} v_0, \quad \bar{w}_0 = \frac{1}{|\Omega|} \int_{\Omega} w_0. \quad (1.6)$$

The conclusions (R1) and (R2) tell us that, when $n \leq 3$, the chemotactic cross-diffusion is substantially overbalanced by random diffusion, and that hence the overall behavior of the model, with respect to both global solvability and asymptotic behavior, is essentially the same as that of the correspondingly modified system without this chemotaxis term. The convergence to the constant equilibrium is uniform as stated in (1.5). While, it is important and curious to ask further questions like: how does the solution converge to that equilibrium, algebraically or exponentially, and, what will happen if $n \geq 4$? The former indeed is the interesting open question left in [4, Remark 4.9] as to their respective rates of convergence in (1.5) except the convergence rate of w . Our primary motivation of this paper is to answer this open question: by carefully utilizing the Neumann heat semigroup theory, we show that any bounded solution of (1.1) converges not only uniformly but also exponentially to its equilibrium $(\bar{u}_0, \bar{v}_0 + \bar{w}_0, 0, \bar{u}_0)$. Moreover, by carefully collecting the appearing constants, we calculate out their explicit rates of convergence in terms of initial datum u_0 and the first nonzero Neumann eigenvalue λ_1 of $-\Delta$ in Ω .

As for the global boundedness, the result (R1) above says that no blow-up phenomenon could occur when $n \leq 3$. It is quite natural to ask what will happen when $n \geq 4$, will

indirect chemotactic cross-diffusion still enforce boundedness unconditionally? To explore this problem, we first observe that the boundedness of the model in the sense of (1.4) can be indeed reduced to the spatial $L^{\frac{n}{2}+\epsilon}$ -boundedness of its u component alone, and then, under certain smallness conditions on the initial data, we demonstrate that infinite chemotactic aggregation can also be fully suppressed. This closes the mathematical completeness of the boundedness for the tumor invasion model (1.1). Our main results read as follows.

Theorem 1.1 (Boundedness and exponential convergence for (1.1)).

- (R3) (**Boundedness criterion in n -D**) Let $n \geq 1$ and let (1.2) hold. Then the boundedness of $\|u(\cdot, t)\|_{L^{\frac{n}{2}+\epsilon}(\Omega)}$ for some $\epsilon > 0$ alone is sufficient to guarantee the boundedness of (1.1) in the sense of (1.4).
- (R4) (**Boundedness in > 3 -D**) Let $n > 3$ and (1.2) hold. Then, one can find $\epsilon_0 > 0$ such that for $\|u_0\|_{L^{\frac{n}{2}}(\Omega)} < \epsilon_0$, $\|z_0\|_{L^{\frac{n}{2}}(\Omega)} < \epsilon_0$ and $\|\nabla v_0\|_{L^n(\Omega)} < \epsilon_0$, the tumor invasion model (1.1) possesses a unique global-in-time classical solution, which is also bounded according to (1.4).
- (R5) (**Exponential Convergence**) Let $n \geq 1$ and $u_0 \geq, \neq 0$. Then all the bounded solutions of (1.1) converge not only uniformly but also exponentially to its equilibrium $(\bar{u}_0, \bar{v}_0 + \bar{w}_0, 0, \bar{u}_0)$. More precisely, for any bounded solution, there exist $t_0 > 0$ and $m_i > 0$ such that

$$\begin{cases} \|u(\cdot, t) - \bar{u}_0\|_{L^\infty(\Omega)} \leq m_1 e^{-\frac{1}{2} \min\{\lambda_1, \frac{\bar{u}_0}{3}\}t}, & \forall t \geq 0, \\ \|v(\cdot, t) - (\bar{v}_0 + \bar{w}_0)\|_{L^\infty(\Omega)} \leq m_2 e^{-\min\{\lambda_1, \frac{\bar{u}_0}{3}\}t}, & \forall t \geq 0, \\ \|w(\cdot, t)\|_{L^\infty(\Omega)} \leq m_3 e^{-\frac{\bar{u}_0}{2}t}, & \forall t \geq 0, \\ \|z(\cdot, t) - \bar{u}_0\|_{L^\infty(\Omega)} \leq m_4 e^{-\min\{1, \frac{\lambda_1}{2}, \frac{\bar{u}_0}{6}\}t}, & \forall t \geq 0. \end{cases} \quad (1.7)$$

Here, $m_i (i = 1, 2, 3, 4)$ are suitably large constants depending on λ_1 , the initial data u_0, v_0, w_0 and Sobolev embedding constants, cf. Section 4.

Thanks to the free mass conversation that $\|u\|_{L^1(\Omega)} = \|u_0\|_{L^1(\Omega)}$, the boundedness criterion (R3) simply recovers (R1) obtained in [4]. This also explains why we always have unconditional boundedness when $n \leq 3$. For the direct Keller-Segel model (1.3), it has been known that the $L^{\frac{n}{2}+\epsilon}$ -boundedness of u implies its L^∞ -boundedness and that certain smallness on the initial data with respect to $L^{\frac{n}{2}}$ -norm of u and L^n -norm of ∇z ensures boundedness [2]. Hence, the criterion (R3) and boundedness (R4) present a mathematical quantification for the tumor model (1.1) as an indirect chemotaxis model compared to the direct chemotaxis model (1.3), in other words, how indirect the tumor model (1.1) is, as a chemotaxis model compared to the direct chemotaxis model (1.3). Besides, we wish to point out that boundedness (R4) imposes no restriction on w_0 . The exponential convergence (R5) surely sharpens the uniform convergence (R2), and therefore resolves the open problem left in [4, Remark 4.9]. Moreover, for the tumor model (1.1), our result shows that boundedness implies not only uniform convergence as shown in [4] but also exponential convergence.

As the direct chemotaxis model (1.3) possesses blow-ups in higher dimensions [17, 18], it would be interesting to investigate whether or not the indirect chemotactic cross-diffusion could drive blow-up phenomenon without smallness conditions on the initial data in higher dimensions ($n \geq 4$)? While, we will leave this for future explorations.

2. PRELIMINARIES

In our subsequent discussions, we need some well-known smoothing $L^p - L^q$ type estimates for the Neumann heat semigroup in Ω . We list them here for convenience, one can find them in [17, Lemma 1.3] and [2, Lemma 2.1].

Lemma 2.1. *Let $(e^{t\Delta})_{t>0}$ be the Neumann heat semigroup in Ω and let $\lambda_1 > 0$ be the first nonzero eigenvalue of $-\Delta$ with homogeneous Neumann boundary condition. Then there exist some constants $k_i (i = 1, 2, 3, 4)$ depending only on Ω such that*

(i) *If $1 \leq q \leq p \leq \infty$, then*

$$\|e^{t\Delta} f\|_{L^p(\Omega)} \leq k_1 \left(1 + t^{-\frac{n}{2}(\frac{1}{q} - \frac{1}{p})}\right) e^{-\lambda_1 t} \|f\|_{L^q(\Omega)} \text{ for all } t > 0 \quad (2.1)$$

holds for all $f \in L^q(\Omega)$ satisfying $\int_{\Omega} f = 0$.

(ii) *If $1 \leq q \leq p \leq \infty$, then*

$$\|\nabla e^{t\Delta} f\|_{L^p(\Omega)} \leq k_2 \left(1 + t^{-\frac{1}{2} - \frac{n}{2}(\frac{1}{q} - \frac{1}{p})}\right) e^{-\lambda_1 t} \|f\|_{L^q(\Omega)} \text{ for all } t > 0 \quad (2.2)$$

is valid for all $f \in L^q(\Omega)$.

(iii) *If $2 \leq q \leq p < \infty$, then*

$$\|\nabla e^{t\Delta} f\|_{L^p(\Omega)} \leq k_3 \left(1 + t^{-\frac{n}{2}(\frac{1}{q} - \frac{1}{p})}\right) e^{-\lambda_1 t} \|\nabla f\|_{L^q(\Omega)} \text{ for all } t > 0 \quad (2.3)$$

is true for all $f \in W^{1,p}(\Omega)$.

(iv) *If $1 < q \leq p \leq \infty$, then*

$$\|e^{t\Delta} \nabla \cdot f\|_{L^p(\Omega)} \leq k_4 \left(1 + t^{-\frac{1}{2} - \frac{n}{2}(\frac{1}{q} - \frac{1}{p})}\right) e^{-\lambda_1 t} \|f\|_{L^q(\Omega)} \text{ for all } t > 0 \quad (2.4)$$

is valid for all $f \in (W^{1,p}(\Omega))^n$.

For convenience, we also write down the local well-posedness of (1.1) and its extendibility criterion from [3, Theorem 3.1] or [4, Lemma 2.1].

Lemma 2.2. *Let $n \geq 1$, $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary and let the initial data (u_0, v_0, w_0, z_0) satisfy (1.2). Then there exist $T_m \in (0, \infty]$ and a unique, nonnegative, classical solution (u, v, w, z) of (1.1) on $\Omega \times [0, T_m)$ such that*

$$\begin{aligned} u &\in C(\bar{\Omega} \times [0, T_m)) \cap C^{2,1}(\bar{\Omega} \times (0, T_m)), \\ v &\in C(\bar{\Omega} \times [0, T_m)) \cap C^{2,1}(\bar{\Omega} \times (0, T_m)) \cap L_{loc}^{\infty}([0, T_m); W^{1,\infty}(\Omega)) \\ w &\in C(\bar{\Omega} \times [0, T_m)) \cap C^{0,1}(\bar{\Omega} \times (0, T_m)), \\ z &\in C(\bar{\Omega} \times [0, T_m)) \cap C^{2,1}(\bar{\Omega} \times (0, T_m)). \end{aligned}$$

Furthermore, if $T_m < \infty$, then

$$\|u(\cdot, t)\|_{L^{\infty}(\Omega)} + \|v(\cdot, t)\|_{W^{1,\infty}(\Omega)} + \|z(\cdot, t)\|_{L^{\infty}(\Omega)} \rightarrow \infty \quad \text{as } t \rightarrow T_m^-. \quad (2.5)$$

The regularity (1.2) will be assumed in force henceforth. The following properties are simple observations from the equations in (1.1) (see [4, Lemmas 2.2, 2.3 and 2.4]).

Lemma 2.3. *The solution (u, v, w, z) of (1.1) enjoys the following properties.*

(i) *The u -component has the mass conservation:*

$$\int_{\Omega} u(x, t) dx = \int_{\Omega} u_0(x) dx, \quad \forall t \in (0, T_m).$$

(ii) *The v - and w -components have the mass conservation that*

$$\int_{\Omega} v(x, t) dx + \int_{\Omega} w(x, t) dx = \int_{\Omega} v_0(x) dx + \int_{\Omega} w_0(x) dx, \quad \forall t \in (0, T_m).$$

(iii) *The w -component is bounded in the way that*

$$\|w(\cdot, t)\|_{L^{\infty}(\Omega)} \leq \|w_0\|_{L^{\infty}(\Omega)}, \quad \forall t \in (0, T_m).$$

In the subsequent text, we shall denote by k_i ($i = 1, 2, 3, 4$) the constants appearing in Lemma 2.1 and by other c_i or C or C generic constants, which may vary from line to line. In most places, we shall write the commonly used short notations like

$$\|f(\cdot, t)\|_{L^p} = \|f(\cdot, t)\|_{L^p(\Omega)} = \left(\int_{\Omega} |f(x, t)|^p dx \right)^{\frac{1}{p}}.$$

3. BOUNDEDNESS IN FOUR AND HIGHER DIMENSIONS

In this section, our goal is to extend the boundedness in ≤ 3 -dimensions of [4] to in ≥ 4 -dimensions under some smallness condition on the initial data. Before proceeding, we first show a boundedness principle which reduces the hard task of proving the L^∞ -boundedness of (1.1) to the less hard task of the $L^{\frac{n}{4}+\epsilon}$ -boundedness of its u component.

Theorem 3.1. *If there exist some $\epsilon > 0$ and $M > 0$ such that*

$$\|u(\cdot, t)\|_{L^{\frac{n}{4}+\epsilon}(\Omega)} \leq M, \quad \forall t \in (0, T_m),$$

then $T_m = \infty$ and the solution (u, v, w, z) of (1.1) is bounded in the sense of (1.4). Moreover, there exist some $\alpha \in (0, 1)$ and constant $C = C(n, \epsilon, M)$ such that

$$\|u\|_{C^{2+\alpha, 1+\frac{\alpha}{2}}(\bar{\Omega} \times [t, t+1])} + \|v\|_{C^{2+\alpha, 1+\frac{\alpha}{2}}(\bar{\Omega} \times [t, t+1])} + \|z\|_{C^{2+\alpha, 1+\frac{\alpha}{2}}(\bar{\Omega} \times [t, t+1])} \leq C, \quad \forall t \geq 1. \quad (3.1)$$

Proof. This is proven via variation-of-constants-formulas from semigroup representation. Its proof will repeatedly utilize [4, Lemmas 2.3, 3.1, 3.2 and 3.4]. To start off, in view of the mass conservation $\|u(\cdot, t)\|_{L^1} = \|u_0\|_{L^1}$, we can take $\epsilon = 1 - \frac{n}{4}$ for $n < 4$. Hence, we shall proceed with $p = \frac{n}{4} + \epsilon \geq 1$, and then from [4, Lemma 3.1] we infer that

$$\|z(\cdot, t)\|_{L^q} \leq C_z(n, \epsilon, q, M), \quad \forall t \in (0, T_m) \quad (3.2)$$

holds for all

$$1 \leq q < \frac{np}{n-2p} \text{ if } p \leq \frac{n}{2}, \quad 1 \leq q \leq \infty \text{ if } p > \frac{n}{2}. \quad (3.3)$$

In the case of $p > \frac{n}{2}$, we take $q = \infty$ in (3.2), and then we see easily from [4, Lemmas 3.2 and 3.3] that $\|\nabla v(\cdot, t)\|_{L^\infty}$ and $\|u(\cdot, t)\|_{L^\infty}$ are uniformly bounded on $(0, T_m)$. Then we are simply done, cf, the end proof of this theorem.

Thus, we will continue with $\frac{n}{4} < p \leq \frac{n}{2}$, which then entails $\frac{np}{n-2p} > \frac{n}{2}$. By Hölder inequality, we can take $q \in (\frac{n}{2}, n)$ in (3.2). Hence, by the semigroup action on the v -equation [4, Lemma 3.2], it follows that

$$\|\nabla v(\cdot, t)\|_{L^r} \leq C_v(n, \epsilon, r, M), \quad \forall t \in (0, T_m) \quad (3.4)$$

holds for all

$$1 \leq r < \frac{nq}{n-q}. \quad (3.5)$$

The choice of $q > \frac{n}{2}$ directly gives $\frac{nq}{n-q} > n$. Hence, one can choose $r > n$ in (3.4) and use [4, Lemma 3.4] to see that u is uniformly bounded:

$$\|u(\cdot, t)\|_{L^\infty} \leq C_u(n, \epsilon, r, q, M), \quad \forall t \in (0, T_m). \quad (3.6)$$

Here, we would like to supply a short proof for (3.6). Indeed, by the variation-of-constants formula for u , we have from (iv) of Lemma 2.1 that

$$\begin{aligned} & \|u(\cdot, t)\|_{L^\infty} \\ & \leq \|e^{t\Delta} u_0\|_{L^\infty} + \int_0^t \|e^{(t-s)\Delta} \nabla \cdot (u(\cdot, s) \nabla v(\cdot, s))\|_{L^\infty} ds \\ & \leq \|u_0\|_{L^\infty} + k_4 \int_0^t (1 + (t-s)^{-\frac{1}{2}-\frac{n}{2\theta}}) e^{-\lambda_1(t-s)} \|u(\cdot, s) \nabla v(\cdot, s)\|_{L^\theta} ds. \end{aligned} \quad (3.7)$$

By taking $\theta \in (\frac{r}{r+1}, r)$, we then infer from Hölder inequality, interpolation inequality, (3.4) and the mass conservation of u , c.f. (i) of Lemma 2.3 that

$$\begin{aligned} \|u(\cdot, s) \nabla v(\cdot, s)\|_{L^\theta} &\leq \|u(\cdot, s)\|_{L^{\frac{r\theta}{r-\theta}}} \|\nabla v(\cdot, s)\|_{L^r} \\ &\leq \|u(\cdot, s)\|_{L^\infty}^{1-\frac{r-\theta}{r\theta}} \|u(\cdot, s)\|_{L^1}^{\frac{r-\theta}{r\theta}} \|\nabla v(\cdot, s)\|_{L^r} \\ &\leq C_v(n, \epsilon, r, M) \|u_0\|_{L^1}^{\frac{r-\theta}{r\theta}} \|u(\cdot, s)\|_{L^\infty}^{1-\frac{r-\theta}{r\theta}}. \end{aligned} \quad (3.8)$$

For any $t \in (0, T_m)$, set $S(t) := \sup_{s \in (0, t)} \|u(\cdot, s)\|_{L^\infty}$. Then, since $S(t)$ is nondecreasing for $t \in (0, T_m)$, we conclude from (3.7) and (3.8) that

$$S(t) \leq \|u_0\|_{L^\infty} + c_1 S^{\frac{r(\theta-1)+\theta}{r\theta}}(t) \int_0^t (1 + (t-s)^{-\frac{1}{2}-\frac{n}{2\theta}}) e^{-\lambda_1(t-s)} ds$$

with $c_1 = k_4 C_v(n, \epsilon, q, M) \|u_0\|_{L^1}^{\frac{r-\theta}{r\theta}}$. It follows from $\theta > n$ the integral $\int_0^t (1 + (t-s)^{-\frac{1}{2}-\frac{n}{2\theta}}) e^{-\lambda_1(t-s)} ds := c_2 < \infty$. Therefore, by Young's inequality, for any $t \in (0, T_m)$,

$$\begin{aligned} S(t) &\leq \|u_0\|_{L^\infty} + c_1 c_2 S(t)^{\frac{r(\theta-1)+\theta}{r\theta}} \leq \|u_0\|_{L^\infty} + \frac{1}{2} S(t) \\ &\quad + \frac{r-\theta}{r\theta} \left[\left(\frac{2r\theta}{(r+1)\theta-r} \right)^{\frac{r\theta}{(r+1)\theta-r}} c_1 c_2 \right]^{\frac{r\theta}{r-\theta}}, \end{aligned}$$

which implies the uniform boundedness of u :

$$\|u(\cdot, t)\|_{L^\infty} \leq S(t) \leq 2 \left\{ \|u_0\|_{L^\infty} + \frac{r-\theta}{r\theta} \left[\left(\frac{2r\theta}{(r+1)\theta-r} \right)^{\frac{r\theta}{(r+1)\theta-r}} c_1 c_2 \right]^{\frac{r\theta}{r-\theta}} \right\}, \quad \forall t \in (0, T_m).$$

Now, it is fairly easy to show the boundedness of $\|v(\cdot, t)\|_{W^{1,\infty}}$ and $\|z(\cdot, t)\|_{L^\infty}$. Twice applications of the maximum principle to the fourth and then second equation in (1.1) deduce $\|z(\cdot, t)\|_{L^\infty} \leq \|u(\cdot, t)\|_{L^\infty}$ and $\|v(\cdot, t)\|_{L^\infty} \leq \|w_0\|_{L^\infty} \|z(\cdot, t)\|_{L^\infty}$, respectively. Hence, by the semigroup action on the v -equation, it follows by taking $q > n$ in [4, Lemma 3.2] that $\|\nabla v(\cdot, t)\|_{L^\infty}$ is uniformly bounded on $(0, T_m)$. These together with the extendibility criterion (2.5) forces $T_m = \infty$. The Hölder regularity (3.1) follows from standard bootstrap arguments involving interior parabolic regularity theory [9]. \square

By the mass conservation of u in (i) of Lemma 2.3, a direct application of Theorem 3.1 recovers the unconditional boundedness for $n \leq 3$ in [4].

Corollary 3.2. *For $n \leq 3$, any solution (u, v, w, z) of the tumor invasion model (1.1) is bounded in the sense of (1.4) and (3.1).*

Next, with the boundedness criterion at hand, we shall show that under some smallness assumptions on the initial data, the classical solution will exist globally with uniform-in-time bound in the case $n \geq 4$.

Theorem 3.3. *Let $n \geq 4$ and the initial data (u_0, v_0, w_0, z_0) satisfy (1.2). If there exists $\varepsilon_0 > 0$ such that*

$$\|u_0\|_{L^{\frac{n}{4}}(\Omega)} \leq \varepsilon, \quad \|z_0\|_{L^{\frac{n}{2}}(\Omega)} \leq \varepsilon \quad \text{and} \quad \|\nabla v_0\|_{L^n(\Omega)} \leq \varepsilon \quad (3.9)$$

for some $\varepsilon < \varepsilon_0$, then the solution (u, v, w, z) of the tumor invasion model (1.1) exists globally in time and is bounded according to (1.4).

Proof. Solving the third equation in (1.1) trivially shows

$$w(x, t) = w_0(x) e^{-\int_0^t z(x, s) ds}.$$

Using the similar argument as in [4, Lemma 4.9] and [16, Lemma 2.9], cf. also the discussion in Lemma 4.3 below, we can find a constant $\delta_1 > 0$ such that $z(t, x) > \delta_1$ for all $x \in \Omega$ and $t > 1$. Hence,

$$\|w\|_{L^\infty} \leq \|w_0\|_{L^\infty} e^{-\delta_1 t}. \quad (3.10)$$

Let $0 < \kappa < \min\{\delta_1, \lambda_1\}$. With $\varepsilon_0 > 0$ to be specified below and assume that (3.9) hold for $\varepsilon \in (0, \varepsilon_0)$. We define

$$T := \sup \left\{ \tilde{T} > 0 \mid \|u(\cdot, t) - e^{t\Delta} u_0\|_{L^\theta} \leq \varepsilon(1 + t^{-2+\frac{n}{2\theta}})e^{-\kappa t} \right. \\ \left. \text{for all } t \in [0, \tilde{T}) \text{ and for all } \theta \in (\frac{n}{4}, \frac{n}{3}) \right\}. \quad (3.11)$$

It is easy to check that T is well defined and positive due to Lemma 2.2. In the sequel, we need to show that $T = \infty$. First, since $n \geq 4$, the smallness condition (3.9) yields

$$\bar{u}_0 = \frac{1}{|\Omega|} \int_{\Omega} u_0 \leq |\Omega|^{-\frac{4}{n}} \|u_0\|_{L^{\frac{n}{4}}} \leq |\Omega|^{-\frac{4}{n}} \varepsilon. \quad (3.12)$$

Then from the definition of T in (3.11) and (3.12), we deduce from Lemma 2.1 (i) that

$$\begin{aligned} \|u(\cdot, t) - \bar{u}_0\|_{L^\theta} &\leq \|u(\cdot, t) - e^{t\Delta} u_0\|_{L^\theta} + \|e^{t\Delta} u_0 - \bar{u}_0\|_{L^\theta} \\ &\leq \varepsilon(1 + t^{-2+\frac{n}{2\theta}})e^{-\kappa t} + k_1(1 + t^{-2+\frac{n}{2\theta}})e^{-\lambda_1 t} \|u_0 - \bar{u}_0\|_{L^{\frac{n}{4}}} \\ &\leq \varepsilon c_1(1 + t^{-2+\frac{n}{2\theta}})e^{-\kappa t}, \quad \forall t \in (0, T), \quad \theta \in (\frac{n}{4}, \frac{n}{3}). \end{aligned} \quad (3.13)$$

From the fourth equation in (1.1) and the mass conservation of u , it follows that

$$(\bar{z})_t = -\bar{z} + \bar{u} = -\bar{z} + \bar{u}_0. \quad (3.14)$$

This joins again the fourth equation in (1.1) entail

$$(z - \bar{z})_t = \Delta(z - \bar{z}) - (z - \bar{z}) + u - \bar{u}_0. \quad (3.15)$$

Then the variation-of-constant formula applied to (3.15) shows that

$$z(\cdot, t) - \bar{z} = e^{t(\Delta-1)}(z_0 - \bar{z}_0) + \int_0^t e^{(t-s)(\Delta-1)}(u(\cdot, s) - \bar{u}_0)ds,$$

which, together with (2.1), (3.9) and (3.13), infers, for $\frac{n}{2} < p < n$, that

$$\begin{aligned} \|z(\cdot, t) - \bar{z}\|_{L^p} &\leq k_1(1 + t^{-1+\frac{n}{2p}})e^{-(\lambda_1+1)t} \|z_0 - \bar{z}_0\|_{L^{\frac{n}{2}}} \\ &\quad + k_1 \int_0^t \left(1 + (t-s)^{-\frac{n}{2}(\frac{1}{\theta}-\frac{1}{p})} \right) e^{-(\lambda_1+1)(t-s)} \|u(\cdot, s) - \bar{u}_0\|_{L^\theta} ds \\ &\leq c_2(1 + t^{-1+\frac{n}{2p}})e^{-(\lambda_1+1)t} \|z_0\|_{L^{\frac{n}{2}}} \\ &\quad + k_1 c_1 \varepsilon \int_0^t \left(1 + (t-s)^{-\frac{n}{2}(\frac{1}{\theta}-\frac{1}{p})} \right) (1 + s^{-2+\frac{n}{2\theta}}) e^{-(\lambda_1+1)(t-s)} e^{-\kappa s} ds \\ &\leq c_2 \varepsilon (1 + t^{-1+\frac{n}{2p}}) e^{-(\lambda_1+1)t} + k_1 c_1 C \varepsilon (1 + t^{\min\{0, -1+\frac{n}{2p}\}}) e^{-\kappa t} \\ &\leq c_3 \varepsilon (1 + t^{-1+\frac{n}{2p}}) e^{-\kappa t}, \end{aligned} \quad (3.16)$$

where we have chosen θ such that $\frac{1}{\theta} = \frac{1}{p} - \delta_2 + \frac{2}{n}$ with $\delta_2 \in (0, \frac{1}{p} - \frac{1}{n})$. In this way, since $p \in (\frac{n}{2}, n)$ we are ensured that $-\frac{n}{2}(\frac{1}{\theta} - \frac{1}{p}) > -1$ as well as $\theta \in (\frac{n}{4}, \frac{n}{3})$. The latter further gives $-2 + \frac{n}{2\theta} > -1$. These in conjunction with $\kappa < \lambda_1$ enable us to employ the following algebraic result [17, Lemma 1.2]:

Lemma 3.4. *Let $\alpha < 1$, $\beta < 1$ and γ, δ be positive constants satisfying $\gamma \neq \delta$. Then there exists some positive constant C depending on $\alpha, \beta, \delta, \gamma$ such that, for all $t > 0$,*

$$\int_0^t (1 + (t-s)^{-\alpha}) e^{-\gamma(t-s)} (1 + s^{-\beta}) e^{-\delta s} ds \leq C(1 + t^{\min\{0, 1-\alpha-\beta\}}) e^{-\min\{\gamma, \delta\}t}.$$

In the next section, at some places, we even need a refined version of such type result and will compute out the constant C explicitly.

By the semigroup representation of the second equation in (1.1) and Lemma 2.1 (ii), for all $q \in (n, \infty)$, we deduce that

$$\begin{aligned}
& \|\nabla(v(\cdot, t) - e^{t\Delta}v_0)\|_{L^q} \\
& \leq \int_0^t \|\nabla e^{(t-s)\Delta}w(z - \bar{z})\|_{L^q} ds + \int_0^t \|\nabla e^{(t-s)\Delta}w\bar{z}\|_{L^q} ds \\
& \leq k_2 \int_0^t \left(1 + (t-s)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\right) e^{-\lambda_1(t-s)} \|w\|_{L^\infty} \|z - \bar{z}\|_{L^p} ds \\
& \quad + k_2 \int_0^t \left(1 + (t-s)^{-\frac{1}{2}}\right) e^{-\lambda_1(t-s)} \|w\|_{L^\infty} \|\bar{z}\|_{L^q} ds \\
& := J_1 + J_2.
\end{aligned} \tag{3.17}$$

For any $q > n$, we first fix $\delta_3 \in (0, \frac{1}{q})$ and then take p satisfying $\frac{1}{p} = \frac{1}{n} + \frac{1}{q} - \delta_3$ so that $-\frac{1}{2} - \frac{n}{2}(\frac{1}{p} - \frac{1}{q}) > -1$ and $\frac{n}{2} < p < n$. Therefore, combining (3.10) and (3.16) and applying Lemma 3.4, we conclude that

$$\begin{aligned}
J_1 & \leq \varepsilon c_3 k_2 \|w_0\|_{L^\infty} \int_0^t \left(1 + (t-s)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\right) (1 + s^{-1+\frac{n}{2p}}) e^{-\lambda_1(t-s)} e^{-\kappa s} ds \\
& \leq \varepsilon c_4 (1 + t^{\min\{0, -\frac{1}{2}+\frac{n}{2q}\}}) e^{-\kappa t} \\
& = \varepsilon c_4 (1 + t^{-\frac{1}{2}+\frac{n}{2q}}) e^{-\kappa t}.
\end{aligned} \tag{3.18}$$

To estimate J_2 , we first solve (3.14) to obtain $\bar{z} = \bar{z}_0 e^{-t} + (1 - e^{-t})\bar{u}_0$, and then

$$\begin{aligned}
\|\bar{z}\|_{L^q} & \leq \|\bar{z}_0 e^{-t}\|_{L^q} + \|(1 - e^{-t})\bar{u}_0\|_{L^q} \\
& = e^{-t} |\Omega|^{-1+\frac{1}{q}} \|z_0\|_{L^1} + (1 - e^{-t}) |\Omega|^{-1+\frac{1}{q}} \|u_0\|_{L^1} \\
& \leq e^{-t} |\Omega|^{\frac{1}{q}-\frac{2}{n}} \|z_0\|_{L^{\frac{n}{2}}} + (1 - e^{-t}) |\Omega|^{\frac{1}{q}-\frac{4}{n}} \|u_0\|_{L^{\frac{n}{4}}} \\
& \leq (|\Omega|^{\frac{1}{q}-\frac{2}{n}} + |\Omega|^{\frac{1}{q}-\frac{4}{n}}) \varepsilon \leq 2 \max\{1, |\Omega|^{-\frac{1}{n}}, |\Omega|^{-\frac{3}{n}}\} \varepsilon.
\end{aligned} \tag{3.19}$$

Using (3.10), (3.19), Lemma 3.4 and noting that $0 < \kappa < \min\{\lambda_1, \delta_1\}$, we have

$$\begin{aligned}
J_2 & \leq 2\varepsilon k_2 \|w_0\|_{L^\infty} \max\{1, |\Omega|^{-\frac{1}{n}}, |\Omega|^{-\frac{3}{n}}\} \int_0^t \left(1 + (t-s)^{-\frac{1}{2}}\right) e^{-\lambda_1(t-s)} e^{-\delta_1 s} ds \\
& \leq \varepsilon c_5 e^{-\min\{\lambda_1, \delta_1\}t} \\
& \leq \varepsilon c_5 (1 + t^{-\frac{1}{2}+\frac{n}{2q}}) e^{-\kappa t}.
\end{aligned} \tag{3.20}$$

A substitution of (3.18) and (3.20) into (3.17) gives rise to

$$\|\nabla(v(\cdot, t) - e^{t\Delta}v_0)\|_{L^q} \leq \varepsilon c_6 (1 + t^{-\frac{1}{2}+\frac{n}{2q}}) e^{-\kappa t}. \tag{3.21}$$

For $q > n \geq 4$, a simple application of (3.9) and (2.3) yields

$$\|\nabla e^{t\Delta}v_0\|_{L^q} \leq k_3 (1 + t^{-\frac{1}{2}+\frac{n}{2q}}) e^{-\lambda_1 t} \|\nabla v_0\|_{L^n} \leq \varepsilon c_7 (1 + t^{-\frac{1}{2}+\frac{n}{2q}}) e^{-\lambda_1 t}. \tag{3.22}$$

Joining (3.21) and (3.22), we arrive at

$$\begin{aligned}
\|\nabla v(\cdot, t)\|_{L^q} & \leq \|\nabla e^{t\Delta}v_0\|_{L^q} + \|\nabla(v(\cdot, t) - e^{t\Delta}v_0)\|_{L^q} \\
& \leq \varepsilon c_7 (1 + t^{-\frac{1}{2}+\frac{n}{2q}}) e^{-\lambda_1 t} + \varepsilon c_6 (1 + t^{-\frac{1}{2}+\frac{n}{2q}}) e^{-\kappa t} \\
& \leq \varepsilon c_8 (1 + t^{-\frac{1}{2}+\frac{n}{2q}}) e^{-\kappa t}, \quad \forall t \in (0, T), \quad q \in (n, \infty).
\end{aligned} \tag{3.23}$$

Now, we apply variation-of-constant formula to the first equation in (1.1) to obtain

$$u(\cdot, t) = e^{t\Delta}u_0 - \int_0^t e^{(t-s)\Delta} \nabla \cdot (u(\cdot, s) \nabla v(\cdot, s)) ds. \quad (3.24)$$

In light of Lemma 2.1 (iv), we infer from (3.24) that

$$\begin{aligned} & \|u(\cdot, t) - e^{t\Delta}u_0\|_{L^\theta} \\ & \leq k_4 \int_0^t \left(1 + (t-s)^{-\frac{1}{2}}\right) e^{-\lambda_1(t-s)} \|u(\cdot, s) \nabla v(\cdot, s)\|_{L^\theta} ds \\ & \leq k_4 \int_0^t \left(1 + (t-s)^{-\frac{1}{2}}\right) e^{-\lambda_1(t-s)} \|(u(\cdot, s) - \bar{u}_0) \nabla v(\cdot, s)\|_{L^\theta} ds \\ & \quad + k_4 \int_0^t \left(1 + (t-s)^{-\frac{1}{2}}\right) e^{-\lambda_1(t-s)} \|\bar{u}_0 \nabla v(\cdot, s)\|_{L^\theta} ds \\ & := I_1 + I_2. \end{aligned} \quad (3.25)$$

Next, we will estimate I_1 and I_2 . As for I_1 , for any $\theta \in (\frac{n}{4}, \frac{n}{3})$ and $\theta_1 \in (\theta, \frac{n}{3})$, we set

$$\theta_2 := \theta_2(\theta, \theta_1) = \frac{\theta\theta_1}{\theta_1 - \theta} \iff \frac{1}{\theta_1} + \frac{1}{\theta_2} = \frac{1}{\theta}.$$

Then it is a little tedious but easy to verify that $\frac{n}{4} < \theta_1 < \frac{n}{3}$, $\theta_2 > \theta_2(\frac{n}{4}, \frac{n}{3}) = n$, $-2 + \frac{n}{2\theta_1} < 0$, $-\frac{1}{2} + \frac{n}{2\theta_2} < 0$ and $-\frac{5}{2} + \frac{n}{2\theta} > -1$. Then from (3.13), (3.23), Lemma 3.4 and Hölder's inequality, we infer that

$$\begin{aligned} I_1 & \leq k_4 \int_0^t \left(1 + (t-s)^{-\frac{1}{2}}\right) e^{-\lambda_1(t-s)} \|u - \bar{u}_0\|_{L^{\theta_1}} \|\nabla v\|_{L^{\theta_2}} ds \\ & \leq c_9 \varepsilon^2 \int_0^t \left(1 + (t-s)^{-\frac{1}{2}}\right) e^{-\lambda_1(t-s)} (1 + s^{-2+\frac{n}{2\theta_1}}) e^{-\kappa s} \cdot (1 + s^{-\frac{1}{2}+\frac{n}{2\theta_2}}) e^{-\kappa s} ds \\ & \leq c_{10} \varepsilon^2 \int_0^t \left(1 + (t-s)^{-\frac{1}{2}}\right) e^{-\lambda_1(t-s)} (1 + s^{-\frac{5}{2}+\frac{n}{2\theta}}) e^{-\kappa s} ds \\ & \leq c_{11} \varepsilon^2 (1 + t^{\min\{0, -2+\frac{n}{2\theta}\}}) e^{-\kappa t} \\ & = c_{11} \varepsilon^2 (1 + t^{-2+\frac{n}{2\theta}}) e^{-\kappa t}. \end{aligned} \quad (3.26)$$

From (3.12) and (3.23), Lemma 3.4 and Hölder's inequality, we control I_2 as follows:

$$\begin{aligned} I_2 & \leq k_4 \int_0^t \left(1 + (t-s)^{-\frac{1}{2}}\right) e^{-\lambda_1(t-s)} \|\bar{u}_0\|_{L^{\theta_1}} \|\nabla v(\cdot, s)\|_{L^{\theta_2}} ds \\ & \leq k_4 c_8 |\Omega|^{\frac{1}{\theta_1} - \frac{4}{n}} \varepsilon^2 \int_0^t \left(1 + (t-s)^{-\frac{1}{2}}\right) e^{-\lambda_1(t-s)} (1 + s^{-\frac{1}{2}+\frac{n}{2\theta_2}}) e^{-\kappa s} ds \\ & \leq k_4 c_8 \max\{|\Omega|^{-\frac{1}{n}}, 1\} \varepsilon^2 \int_0^t \left(1 + (t-s)^{-\frac{1}{2}}\right) e^{-\lambda_1(t-s)} (1 + s^{-2+\frac{n}{2\theta_1}}) \\ & \quad \times (1 + s^{-\frac{1}{2}+\frac{n}{2\theta_2}}) e^{-\kappa s} ds \\ & \leq c_{12} \varepsilon^2 \int_0^t \left(1 + (t-s)^{-\frac{1}{2}}\right) e^{-\lambda_1(t-s)} (1 + s^{-\frac{5}{2}+\frac{n}{2\theta}}) e^{-\kappa s} ds \\ & \leq c_{13} \varepsilon^2 (1 + t^{-2+\frac{n}{2\theta}}) e^{-\kappa t}. \end{aligned} \quad (3.27)$$

Substituting (3.26) and (3.27) into (3.25), we finally obtain the crucial estimate for u :

$$\|u(\cdot, t) - e^{t\Delta}u_0\|_{L^\theta} \leq c_{14} \varepsilon^2 (1 + t^{-2+\frac{n}{2\theta}}) e^{-\kappa t}. \quad (3.28)$$

Hence, upon choosing $\varepsilon_0 < \frac{1}{2c_{14}}$ in (3.28), we argue by contraction from the definition of T in (3.11) that $T = \infty$. Then the fact that $T \leq T_m$ (the maximal existence time) directly

concludes that $T_m = \infty$, and so (3.12) and (3.13) enable us to obtain that

$$\begin{aligned} \|u(\cdot, t)\|_{L^\theta} &\leq \|u(\cdot, t) - \bar{u}_0\|_{L^\theta} + \|\bar{u}_0\|_{L^\theta} \\ &\leq 2\varepsilon c_1 e^{-\kappa t} + |\Omega|^{\frac{1}{\theta} - \frac{4}{n}} \varepsilon \leq (2c_1 + \max\{|\Omega|^{-\frac{1}{n}}, 1\})\varepsilon < \infty \end{aligned} \quad (3.29)$$

for all $t \geq 1$. While, for $t < 1$, the result of local existence guarantees $\|u(\cdot, t)\|_{L^\theta} < \infty$. Then the uniform boundedness of $\|u(\cdot, t)\|_{L^\theta}$ with $\theta > \frac{n}{4}$ together with Theorem 3.1 implies the boundedness of solutions (u, v, w, z) as specified in the theorem. \square

4. BOUNDEDNESS IMPLIES EXPONENTIAL CONVERGENCE

4.1. Boundedness implies convergence. To our further purpose, we here observe that the discussions in [4] show that the boundedness of (1.1) indeed implies its convergence. Indeed, by integrating the third equation in (1.1) over $\Omega \times (0, t)$ and recalling that w is bounded, cf. Lemma 2.3 (iii), we get the following observation:

Lemma 4.1 ([4]). *The solution of (1.1) has the property that*

$$\int_0^\infty \int_\Omega w(x, t) z(x, t) dx dt < \infty.$$

Combing this crucial property with the parabolic regularity (3.1), we can conclude the uniform convergence of solutions as argued in [4].

Lemma 4.2. *Any bounded solution (u, v, w, z) of (1.1) will converge according to*

$$\left\| (u(\cdot, t), v(\cdot, t), w(\cdot, t), z(\cdot, t)) - (\bar{u}_0, \bar{v}_0 + \bar{w}_0, 0, \bar{u}_0) \right\|_{L^\infty(\Omega)} \rightarrow 0, \quad (4.1)$$

as $t \rightarrow \infty$, where \bar{u}_0, \bar{v}_0 and \bar{w}_0 are defined by (1.6).

4.2. Exponential convergences and their convergence rates. In this subsection, we will show that the bounded solution of (1.1) converges not only uniformly but also exponentially. Beyond that, by carefully collecting the appearing constants, we shall calculate out their explicit rates of convergence in terms of initial datum u_0 and the first nonzero Neumann eigenvalue λ_1 of $-\Delta$ in Ω . Hereafter, we shall assume that the solution quadruple (u, v, w, z) of (1.1) is bounded and so we have the convergence (4.1). Also, the condition $u_0 \geq, \neq 0$ will be assumed in order to avoid saying something trivial. Thanks to (4.1), we henceforth fix a $t_0 \geq 0$ such that

$$\frac{\bar{u}_0}{2} \leq u, z \leq \frac{3\bar{u}_0}{2}, \quad \frac{(\bar{v}_0 + \bar{w}_0)}{2} \leq v \leq \frac{3(\bar{v}_0 + \bar{w}_0)}{2} \quad \text{on } \bar{\Omega} \times [t_0, \infty). \quad (4.2)$$

4.2.1. Convergence rate of w .

Lemma 4.3. *The solution component w converges exponentially to 0:*

$$\|w(\cdot, t)\|_{L^\infty(\Omega)} \leq \|w_0\|_{L^\infty(\Omega)} e^{-\frac{\bar{u}_0}{2}(t-t_0)}, \quad \forall t \geq t_0. \quad (4.3)$$

Proof. Solving the third equation in (1.1), we get

$$w(x, t) = w(x, t_0) e^{-\int_{t_0}^t z(x, s) ds},$$

which, coupled with the boundedness of w in Lemma 2.3 (iii) and the fact that $z \geq \frac{\bar{u}_0}{2}$ on $\bar{\Omega} \times [t_0, \infty)$, c.f. (4.2), trivially leads to the desired estimate (4.3). \square

4.2.2. *Convergence rate of v .* In this subsection, we will compute the exponential convergence rate of v . In fact, we obtain the following explicit decaying estimate for v .

Lemma 4.4. *The solution component v has the following decay rate: for all $t \geq t_0$,*

$$\|v(\cdot, t) - (\bar{v}_0 + \bar{w}_0)\|_{L^\infty(\Omega)} \leq \left[6k_1(\bar{v}_0 + \bar{w}_0) + \left(\frac{36k_1}{e} + 1 \right) \|w_0\|_{L^\infty(\Omega)} \right] e^{-\min\{\lambda_1, \frac{\bar{u}_0}{3}\}(t-t_0)}. \quad (4.4)$$

Proof. From the second equation in (1.1), we get

$$(v - \bar{v})_t = \Delta(v - \bar{v}) + wz - \overline{wz}, \quad (4.5)$$

by noting that

$$\frac{d}{dt}\bar{v} = \overline{wz}.$$

The variation-of-constant formula applied to (4.5) leads to

$$v(\cdot, t) - \bar{v} = e^{(t-t_0)\Delta} (v(\cdot, t_0) - \bar{v}(t_0)) + \int_{t_0}^t e^{(t-s)\Delta} (wz - \overline{wz}) ds, \quad t \geq t_0,$$

which, together with (2.1), (4.2) and (4.3), gives

$$\begin{aligned} & \|v(\cdot, t) - \bar{v}\|_{L^\infty} \\ & \leq \|e^{(t-t_0)\Delta} (v(\cdot, t_0) - \bar{v}(t_0))\|_{L^\infty} + \int_{t_0}^t \|e^{(t-s)\Delta} (wz - \overline{wz})\|_{L^\infty} ds \\ & \leq 2k_1 \|v(\cdot, t_0) - \bar{v}(t_0)\|_{L^\infty} e^{-\lambda_1(t-t_0)} + 2k_1 \int_{t_0}^t e^{-(t-s)\lambda_1} \|(wz)(s) - \overline{wz}(s)\|_{L^\infty} ds \\ & \leq 6k_1(\bar{v}_0 + \bar{w}_0) e^{-\lambda_1(t-t_0)} + 6\bar{u}_0 k_1 \|w_0\|_{L^\infty} \int_{t_0}^t e^{-(t-s)\lambda_1} e^{-\frac{\bar{u}_0}{2}(s-t_0)} ds \\ & \leq 6k_1(\bar{v}_0 + \bar{w}_0) e^{-\lambda_1(t-t_0)} + \frac{36}{e} k_1 \|w_0\|_{L^\infty} e^{-\frac{\bar{u}_0}{3}(t-t_0)} \\ & \leq 6k_1 \left[(\bar{v}_0 + \bar{w}_0) + \frac{6}{e} \|w_0\|_{L^\infty} \right] e^{-\min\{\lambda_1, \frac{\bar{u}_0}{3}\}(t-t_0)}, \end{aligned} \quad (4.6)$$

where we have used the following algebraic calculations:

$$\begin{aligned} \int_{t_0}^t e^{-(t-s)\lambda_1} e^{-\frac{\bar{u}_0}{2}(s-t_0)} ds &= \begin{cases} (t-t_0) e^{-\lambda_1(t-t_0)}, & \text{if } \lambda_1 = \frac{\bar{u}_0}{2} \\ \frac{1}{\lambda_1 - \frac{\bar{u}_0}{2}} \left[e^{-\lambda_1(t-t_0)} - e^{-\frac{\bar{u}_0}{2}(t-t_0)} \right], & \text{if } \lambda_1 \neq \frac{\bar{u}_0}{2} \end{cases} \\ &\leq (t-t_0) e^{-\frac{\bar{u}_0}{2}(t-t_0)} \leq \frac{6}{\bar{u}_0 e} e^{-\frac{\bar{u}_0}{3}(t-t_0)}, \quad \forall t \geq t_0. \end{aligned}$$

On the other hand, by the mass conservation of $v + w$ in Lemma 2.3 (ii) and (4.3) of Lemma 4.3, we deduce

$$\|\bar{v} - (\bar{v}_0 + \bar{w}_0)\|_{L^\infty} = \|\bar{v} - \bar{w}\|_{L^\infty(\Omega)} \leq \|w\|_{L^\infty} \leq \|w_0\|_{L^\infty} e^{-\frac{\bar{u}_0}{2}(t-t_0)},$$

which together with (4.6) leads to (4.4). \square

4.2.3. *Convergence rate of u .* In this subsection, we shall illustrate that u stabilizes exponentially to its spatial mean. To this end, a decay rate of ∇v in $L^p(\Omega)$ for arbitrary $p > 2$ is needed and is essential in our subsequent analyses.

Lemma 4.5. *For any $2 < p < \infty$, the gradient of v has the following decay estimate:*

$$\|\nabla v(\cdot, t)\|_{L^p(\Omega)} \leq C e^{-\frac{1}{2} \min\{\lambda_1, \frac{\bar{u}_0}{3}\}(t-t_0)}, \quad \forall t \geq t_0, \quad (4.7)$$

where

$$C = C(\|\nabla v(\cdot, t_0)\|_{L^p}, \bar{u}_0, \lambda_1) = \left[2k_3 \|\nabla v(\cdot, t_0)\|_{L^p} + \frac{3}{2} k_2 \bar{u}_0 \|w_0\|_{L^\infty} |\Omega|^{\frac{1}{p}} \max\{A, B\} \right] \quad (4.8)$$

with $A = A(\bar{u}_0, \lambda_1)$ and $B = B(\bar{u}_0)$ defined respectively by (4.12) and (4.13) below.

Proof. Applying the variation of constants representation to the second equation in (1.1), then, for all $t \geq t_0$, we have

$$v(\cdot, t) = e^{(t-t_0)\Delta}v(\cdot, t_0) - \int_{t_0}^t e^{(t-s)\Delta}w(\cdot, s)z(\cdot, s)ds,$$

which implies

$$\|\nabla v(\cdot, t)\|_{L^p} \leq \|\nabla e^{(t-t_0)\Delta}v(\cdot, t_0)\|_{L^p} + \int_{t_0}^t \|\nabla e^{(t-s)\Delta}w(\cdot, s)z(\cdot, s)\|_{L^p} ds. \quad (4.9)$$

For $p > 2$, we use the heat semigroup property (2.3) to derive

$$\|\nabla e^{(t-t_0)\Delta}v(\cdot, t_0)\|_{L^p} \leq 2k_3 \|\nabla v(\cdot, t_0)\|_{L^p} e^{-\lambda_1(t-t_0)}, \quad \forall t \geq t_0. \quad (4.10)$$

Moreover, using (2.2) and the Hölder inequality, one infers from (4.2) and (4.3) that

$$\begin{aligned} & \int_{t_0}^t \|\nabla e^{(t-s)\Delta}w(\cdot, s)z(\cdot, s)\|_{L^p} ds \\ & \leq k_2 \int_{t_0}^t \left(1 + (t-s)^{-\frac{1}{2}}\right) e^{-\lambda_1(t-s)} \|w(\cdot, s)z(\cdot, s)\|_{L^p} ds \\ & \leq k_2 |\Omega|^{\frac{1}{p}} \int_{t_0}^t \left(1 + (t-s)^{-\frac{1}{2}}\right) e^{-\lambda_1(t-s)} \|w(\cdot, s)\|_{L^\infty} \|z(\cdot, s)\|_{L^\infty} ds \\ & \leq \frac{3}{2} k_2 \bar{u}_0 \|w_0\|_{L^\infty} |\Omega|^{\frac{1}{p}} \int_{t_0}^t \left(1 + (t-s)^{-\frac{1}{2}}\right) e^{-\lambda_1(t-s)} e^{-\frac{\bar{u}_0}{2}(s-t_0)} ds \\ & = \frac{3}{2} k_2 \bar{u}_0 \|w_0\|_{L^\infty} |\Omega|^{\frac{1}{p}} e^{-\frac{\bar{u}_0}{2}(t-t_0)} \int_0^{t-t_0} \left(1 + \tau^{-\frac{1}{2}}\right) e^{-(\lambda_1 - \frac{\bar{u}_0}{2})\tau} d\tau. \end{aligned} \quad (4.11)$$

By straightforward computations, we have

$$\begin{aligned} \int_0^{t-t_0} \left(1 + \tau^{-\frac{1}{2}}\right) e^{-(\lambda_1 - \frac{\bar{u}_0}{2})\tau} d\tau & \leq \begin{cases} (t-t_0) + 2(t-t_0)^{\frac{1}{2}}, & \text{if } \lambda_1 \geq \frac{\bar{u}_0}{2}, \\ Ae^{(\frac{\bar{u}_0}{2} - \frac{\lambda_1}{2})(t-t_0)}, & \text{if } \lambda_1 < \frac{\bar{u}_0}{2}, \end{cases} \\ & \leq \begin{cases} Be^{\frac{\bar{u}_0}{3}(t-t_0)}, & \text{if } \lambda_1 \geq \frac{\bar{u}_0}{2}, \\ Ae^{(\frac{\bar{u}_0}{2} - \frac{\lambda_1}{2})(t-t_0)}, & \text{if } \lambda_1 < \frac{\bar{u}_0}{2}, \end{cases} \end{aligned}$$

with

$$A = A(\bar{u}_0, \lambda_1) = \sup_{s>0} \left\{ e^{-(\frac{\bar{u}_0}{2} - \frac{\lambda_1}{2})s} \int_0^s \left(1 + \tau^{-\frac{1}{2}}\right) e^{-(\lambda_1 - \frac{\bar{u}_0}{2})\tau} d\tau \right\} < \infty \quad (4.12)$$

and

$$B = B(\bar{u}_0) = \sup_{s>0} \left\{ (s + 2s^{\frac{1}{2}}) e^{-\frac{\bar{u}_0}{3}s} \right\} < \infty. \quad (4.13)$$

Then inserting these inequalities into (4.11), we derive

$$\begin{aligned} & \int_{t_0}^t \|\nabla e^{(t-s)\Delta}w(\cdot, s)z(\cdot, s)\|_{L^p} ds \\ & \leq \frac{3}{2} k_2 \bar{u}_0 \|w_0\|_{L^\infty} |\Omega|^{\frac{1}{p}} \max\{A, B\} e^{-\frac{1}{2} \min\{\lambda_1, \frac{\bar{u}_0}{3}\}(t-t_0)}. \end{aligned} \quad (4.14)$$

Substituting (4.14) and (4.10) into (4.9), we end up with (4.7). \square

Now, we are well prepared to show the rate of exponential convergence of u .

Lemma 4.6. *The u -component has an exponential decay estimate that*

$$\|u(\cdot, t) - \bar{u}_0\|_{L^\infty(\Omega)} \leq \left(5k_1 + \frac{3}{2}k_4CD\right) \bar{u}_0 e^{-\frac{1}{2} \min\{\lambda_1, \frac{\bar{u}_0}{3}\}(t-t_0)}, \quad \forall t \geq t_0, \quad (4.15)$$

where C and D are defined by (4.8) and (4.20), respectively.

Proof. It follows from the first equation in (1.1) that

$$(u - \bar{u})_t = \Delta(u - \bar{u}) - \nabla \cdot (u \nabla v). \quad (4.16)$$

The variation-of-constant formula then shows from (4.16) that

$$u(\cdot, t) - \bar{u} = e^{(t-t_0)\Delta}(u(\cdot, t_0) - \bar{u}(t_0)) - \int_{t_0}^t e^{(t-s)\Delta} \nabla \cdot (u(\cdot, s) \nabla v(\cdot, s)) ds,$$

which gives rise to

$$\begin{aligned} \|u(\cdot, t) - \bar{u}\|_{L^\infty} &\leq \|e^{(t-t_0)\Delta}(u(\cdot, t_0) - \bar{u}(t_0))\|_{L^\infty} \\ &\quad + \int_{t_0}^t \|e^{(t-s)\Delta} \nabla \cdot (u(\cdot, s) \nabla v(\cdot, s))\|_{L^\infty} ds. \end{aligned} \quad (4.17)$$

Using (2.1) and noting that $\int_\Omega (u - \bar{u}) = 0$ and $\bar{u} = \bar{u}_0$, one has from (4.2) that

$$\|e^{(t-t_0)\Delta}(u(\cdot, t_0) - \bar{u})\|_{L^\infty} \leq 2k_1 \|u(\cdot, t_0) - \bar{u}_0\|_{L^\infty} e^{-\lambda_1(t-t_0)} \leq 6k_1 \bar{u}_0 e^{-\lambda_1(t-t_0)}. \quad (4.18)$$

Furthermore, according to (2.4), (4.2) and Lemma 4.5 with $p = 3n > 2$, we deduce

$$\begin{aligned} &\int_{t_0}^t \|e^{(t-s)\Delta} \nabla \cdot (u(\cdot, s) \nabla v(\cdot, s))\|_{L^\infty} ds \\ &\leq k_4 \int_{t_0}^t \left(1 + (t-s)^{-\frac{1}{2} - \frac{n}{2*3n}}\right) e^{-\lambda_1(t-s)} \|u(\cdot, s)\|_{L^\infty} \|\nabla v(\cdot, s)\|_{L^{3n}} ds \\ &\leq \frac{3\bar{u}_0}{2} k_4 C \int_{t_0}^t \left(1 + (t-s)^{-\frac{2}{3}}\right) e^{-\lambda_1(t-s)} e^{-\frac{1}{2} \min\{\lambda_1, \frac{\bar{u}_0}{3}\}(s-t_0)} ds \\ &= \frac{3\bar{u}_0}{2} k_4 C e^{-\frac{1}{2} \min\{\lambda_1, \frac{\bar{u}_0}{3}\}(t-t_0)} \int_0^{t-t_0} (1 + \tau^{-\frac{2}{3}}) e^{-(\lambda_1 - \frac{1}{2} \min\{\lambda_1, \frac{\bar{u}_0}{3}\})\tau} d\tau \\ &\leq \frac{3\bar{u}_0}{2} k_4 C D e^{-\frac{1}{2} \min\{\lambda_1, \frac{\bar{u}_0}{3}\}(t-t_0)} \end{aligned} \quad (4.19)$$

where

$$D = D(\bar{u}_0, \lambda_1) = \sup_{s>0} \int_0^s (1 + \tau^{-\frac{2}{3}}) e^{-(\lambda_1 - \frac{1}{2} \min\{\lambda_1, \frac{\bar{u}_0}{3}\})\tau} d\tau < \infty. \quad (4.20)$$

Substituting (4.18) and (4.19) into (4.17), we obtain (4.15). \square

4.2.4. *Convergence of z .* Finally, we compute the rate of exponential convergence of z .

Lemma 4.7. *The z -component has the following exponential decay estimate:*

$$\|z(\cdot, t) - \bar{u}_0\|_{L^\infty(\Omega)} \leq \bar{u}_0 \left\{ \frac{5}{2} + 2k_1 \left[3 + \frac{(10k_1 + 3k_4 CD)}{2(\lambda_1 + 1) - \min\{\lambda_1, \frac{\bar{u}_0}{3}\}} \right] \right\} e^{-\min\{1, \frac{\lambda_1}{2}, \frac{\bar{u}_0}{6}\}(t-t_0)} \quad (4.21)$$

for all $t \geq t_0$, where C and D are given by (4.8) and (4.20).

Proof. Notice from the fourth equation in (1.1) that $(\bar{z})_t = -\bar{z} + \bar{u} = -\bar{z} + \bar{u}_0$, we have

$$(z - \bar{z})_t = \Delta(z - \bar{z}) - (z - \bar{z}) + u - \bar{u}_0. \quad (4.22)$$

The variation-of-constant formula applied to (4.22) gives that

$$z(\cdot, t) - \bar{z} = e^{(t-t_0)(\Delta-1)}(z(\cdot, t_0) - \bar{z}(t_0)) + \int_{t_0}^t e^{(t-s)(\Delta-1)}(u(\cdot, s) - \bar{u}_0) ds. \quad (4.23)$$

Now, it follows from (2.1) and (4.2) that

$$\begin{aligned} \|e^{(t-t_0)(\Delta-1)}(z(\cdot, t_0) - \bar{z}(t_0))\|_{L^\infty} &\leq 2k_1 e^{-(\lambda_1+1)(t-t_0)} \|z(\cdot, t_0) - \bar{z}(t_0)\|_{L^\infty} \\ &\leq 6k_1 \bar{u}_0 e^{-(\lambda_1+1)(t-t_0)} \end{aligned} \quad (4.24)$$

and, thanks to (4.15),

$$\begin{aligned}
& \left\| \int_{t_0}^t e^{(t-s)(\Delta-1)} (u(\cdot, s) - \bar{u}_0) ds \right\|_{L^\infty} \\
& \leq 2k_1 \int_{t_0}^t e^{-(\lambda_1+1)(t-s)} \|u(\cdot, s) - \bar{u}_0\|_{L^\infty} ds \\
& \leq 2k_1 (5k_1 + \frac{3}{2}k_4 CD) \bar{u}_0 e^{-\frac{1}{2} \min\{\lambda_1, \frac{\bar{u}_0}{3}\}(t-t_0)} \int_0^{t-t_0} e^{-[(\lambda_1+1) - \frac{1}{2} \min\{\lambda_1, \frac{\bar{u}_0}{3}\}]s} ds \\
& \leq \frac{2k_1 (10k_1 + 3k_4 CD) \bar{u}_0}{2(\lambda_1 + 1) - \min\{\lambda_1, \frac{\bar{u}_0}{3}\}} e^{-\frac{1}{2} \min\{\lambda_1, \frac{\bar{u}_0}{3}\}(t-t_0)}.
\end{aligned} \tag{4.25}$$

By taking L^∞ -norm on both sides of (4.23) and using (4.24) and (4.25), we arrive at

$$\|z(\cdot, t) - \bar{z}\|_{L^\infty} \leq 2k_1 \bar{u}_0 \left[3 + \frac{(10k_1 + 3k_4 CD)}{2(\lambda_1 + 1) - \min\{\lambda_1, \frac{\bar{u}_0}{3}\}} \right] e^{-\frac{1}{2} \min\{\lambda_1, \frac{\bar{u}_0}{3}\}(t-t_0)}, \forall t \geq t_0. \tag{4.26}$$

On the other hand, we note from $(\bar{z})_t = -\bar{z} + \bar{u}$ and $\bar{u} = \bar{u}_0$ that

$$\bar{z}(t) - \bar{u}_0 = [\bar{z}(t_0) - \bar{u}_0] e^{-(t-t_0)} + e^{-t} \int_{t_0}^t (\bar{u}(s) - \bar{u}_0) e^s ds = [\bar{z}(t_0) - \bar{u}_0] e^{-(t-t_0)}.$$

Then (4.2) implies

$$\|\bar{z}(t) - \bar{u}_0\|_{L^\infty} \leq \|\bar{z}(t_0) - \bar{u}_0\|_{L^\infty} e^{-(t-t_0)} \leq \frac{5\bar{u}_0}{2} e^{-(t-t_0)}.$$

This combined with (4.26) gives the desired estimate (4.21). \square

The main exponential decay estimates in (1.7) of Theorem 1.1 follow a collection of Lemmas 4.3 -4.7 with perhaps some large constants m_i . At the end of this paper, we remark that Lemma 4.7 may be shown alternatively via comparison principle.

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REFERENCES

- [1] M. Chaplain and A. Anderson, Mathematical modelling of tissue invasion, Cancer modelling and simulation, Chapman & Hall/CRC Math. Biol. Med. Ser., Chapman & Hall/CRC, Boca Raton, FL, (2003), 269–297.
- [2] X. Cao, Global bounded solutions of the higher-dimensional Keller-Segel system under smallness conditions in optimal spaces, Discrete Contin. Dyn. Syst. 35 (2015), 1891–1904.
- [3] K. Fujie, A. Ito and T. Yokota, Existence and uniqueness of local classical solutions to modified tumor invasion models of Chaplain-Anderson type, Adv. Math. Sci. Appl. 24 (2014), 67–84.
- [4] K. Fujie, A. Ito, Y. Winkler and T. Yokota, Stabilization in a chemotaxis model for tumor invasion, Discrete Contin. Dyn. Syst. 36 (2016), 151–169.
- [5] M. Herrero and J. Velázquez, A blow-up mechanism for a chemotaxis model, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 24 (1997), 633–683.
- [6] T. Hillen and A. Potapov, The one-dimensional chemotaxis model: global existence and asymptotic profile, Math. Methods Appl. Sci. 27 (2004), 1783–1801.
- [7] W. Jäger and S. Luckhaus, On explosions of solutions to a system of partial differential equations modeling chemotaxis, Trans. Amer. Math. Soc. 329 (1992), 819–824.
- [8] E. Keller and L. Segel, Initiation of slime mold aggregation viewed as an instability, J. Theoret. Biol., 26 (1970), 399–415.
- [9] O. Ladyzhenskaya, V. Solonnikov and N. Uralceva, Linear and Quasilinear Equations of Parabolic Type, AMS, Providence, RI, 1968.
- [10] N. Mizoguchi and M. Winkler, Blow-up in the two-dimensional Keller-Segel system, preprint.

- [11] T. Nagai, Blow-up of radially symmetric solutions to a chemotaxis system. *Adv. Math. Sci. Appl.* 5 (1995), 581–601.
- [12] T. Nagai, T. Senba and K. Yoshida, Application of the Trudinger-Moser inequality to a parabolic system of chemotaxis, *Funkcial. Ekvac.* 40 (1997), 411–433.
- [13] K. Osaki and A. Yagi, Finite dimensional attractor for one-dimensional Keller-Segel equations, *Funkcial. Ekvac.* 44 (2001), 441–469.
- [14] T. Senba and T. Suzuki, Parabolic system of chemotaxis: blowup in a finite and the infinite time, *Methods Appl. Anal.* 8 (2001), 349–367.
- [15] Y. Tao and M. Winkler, Boundedness in a quasilinear parabolic-parabolic Keller-Segel system with subcritical sensitivity, *J. Differential Equations* 252 (2012), 692–715.
- [16] Y. Tao and M. Winkler, Large Time Behavior in a Multidimensional Chemotaxis-Haptotaxis Model with Slow Signal Diffusion. *SIAM J. Math. Anal.* 47(6) (2015), 4229–4250.
- [17] M. Winkler, Aggregation vs. global diffusive behavior in the higher-dimensional Keller-Segel model, *J. Differential Equations* 248 (2010), 2889–2905.
- [18] M. Winkler, Finite-time blow-up in the higher-dimensional parabolic-parabolic Keller-Segel system, *J. Math. Pures Appl.* 100 (2013), 748–767.

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